Math 534 - Exercise Sheet 1

0. Prove all the blue claims in the lecture.

1. Show that if a bialgebra A admits an antipode $S : A \to A$ which satisfies the axioms of a Hopf algebra, then such an S is unique (in other words, the existence of an antipode is a property and not an extra structure of a bialgebra).

2. For a finite group G, construct a natural Hopf algebra structure on the group algebra $\mathbb{C}G$.

3. Consider the algebra A generated by g, x subject to the relations

$$g^2 = 1, \qquad x^2 = 0, \qquad gx + xg = 0$$

Calculate the dimension of A and endow it with a bialgebra structure such that $\Delta(g) = g \otimes g$. Prove that it is actually a Hopf algebra. (*This is the smallest dimensional Hopf algebra which is neither commutative nor cocommutative*).

4. If A is an infinite-dimensional algebra, then its **restricted dual** is

$$A^* = \left\{ \lambda : A \to \mathbb{C}^* \;\middle|\; \exists \text{ideal } I \subset \text{Ker } \lambda \text{ s.t. } \dim_{\mathbb{C}}(A/I) < \infty \right\}$$

Show that A^* is a coalgebra, and show that if A is a bialgebra then so is A^*.